

# Spectrum structure for eigenvalue problems involving mean curvature operators in Euclidean and Minkowski spaces \*

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## Abstract

In this paper, we are concerned with quasilinear Dirichlet problem

$$\begin{cases} -\left(\frac{u'(x)}{\sqrt{1+\kappa(u'(x))^2}}\right)' = \lambda u(x), & 0 < x < 1, \\ u(0) = u(1) = 0, \end{cases} \quad (P)$$

where  $\kappa \in (-\infty, 0) \cup (0, \infty)$  is a constant. We show that any nontrivial solution  $u$  of (P) has only finite many of simple zeros in  $[0, 1]$ , all of humps of  $u$  are same, and the first hump is symmetric around the middle point of its domain. We also describe the global structure of the set of nontrivial solutions of (P).

**Keywords.** One-dimensional Euclidean-curvature operator, one-dimensional Minkowski-curvature operator, eigenvalue, sign-changing solutions.

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## 1 Introduction

This paper studies the *spectrum structure* of the quasilinear Dirichlet problem

$$\begin{cases} -\left(\frac{u'(x)}{\sqrt{1+\kappa(u'(x))^2}}\right)' = \lambda u(x), & 0 < x < 1, \\ u(0) = u(1) = 0, \end{cases} \quad (1.1)$$

where

$$' = D := \frac{d}{dx},$$

and  $\kappa \neq 0$  is a constant. This problem establishes a quasilinear continuum deformation between the linear eigenvalue problem

$$\begin{cases} -u''(x) = \lambda u(x), & 0 < x < 1, \\ u(0) = u(1) = 0, \end{cases} \quad (1.2)$$

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and (1.1), which is a quasilinear problem associated to the one-dimensional Euclidean-curvature operator as  $\kappa > 0$  and to the one-dimensional Minkowski-curvature operator as  $\kappa < 0$ , respectively.

The quasilinear problem (1.1) can be equivalently written as

$$\begin{cases} -\frac{u''(x)}{[1 + \kappa(u'(x))^2]^{3/2}} = \lambda u(x), & 0 < x < 1, \\ u(0) = u(1) = 0, \end{cases} \quad (1.3)$$

or

$$\begin{cases} -u''(x) = \lambda u[1 + \kappa(u'(x))^2]^{3/2}, & 0 < x < 1, \\ u(0) = u(1) = 0. \end{cases} \quad (1.4)$$

Cano-Casanova, López-Gómez and Takimoto [1] used the standard techniques from bifurcation theory [2-5] to study the existence of positive solutions for (1.4) with  $\kappa > 0$ . Their main results can be summarized in the following list:

- Problem (1.4) has a positive solution if and only if

$$8B^2 < \lambda < \pi^2, \quad B = \int_0^1 \frac{d\theta}{\sqrt{\theta^{-4} - 1}}. \quad (1.5)$$

- The positive solution of (1.4) is unique if it exists. Subsequently, we denote it by  $u_\lambda$ .
- $u_\lambda$  is symmetric around  $1/2$  for all  $\lambda$  satisfying (1.5).
- $u'_\lambda(0)$  is strictly decreasing in  $(8B^2, \pi^2)$ .
- $u_\lambda$  satisfies

$$\lim_{\lambda \downarrow 8B^2} u'_\lambda(0) = \infty, \quad \lim_{\lambda \downarrow 8B^2} \|u_\lambda\|_\infty = \frac{1}{2B\sqrt{\kappa}}, \quad (1.6)$$

and

$$\lim_{\lambda \uparrow \pi^2} \|u_\lambda\|_\infty = \lim_{\lambda \uparrow \pi^2} u_\lambda(1/2) = 0. \quad (1.7)$$

- The point-wise limit  $u_{8B^2} := \lim_{\lambda \downarrow 8B^2} u_\lambda$  satisfies

$$u'_{8B^2}(0) = -u'_{8B^2}(1) = \infty. \quad (1.8)$$

Their results are motivated by the pioneering results in Nakao [6] and have been extended to the more general case

$$\begin{cases} -\left(\frac{u'(x)}{\sqrt{1 + \kappa(u'(x))^2}}\right)' = \lambda V(x)u(x), & 0 < x < 1, \\ u(0) = u(1) = 0, \end{cases}$$

by Cano-Casanova, López-Gómez and Takimoto [7], where the weight function  $V(x) \geq 0$  and  $V(x) \not\equiv 0$  in  $[0, 1]$ .

It is well-known that (1.2) has a sequence of eigenvalues

$$\lambda_n = n^2 \pi^2, \quad n \in \{1, 2, \dots\}.$$

For each  $n \in \mathbb{N}$ ,  $\lambda_n$  is simple, and its eigenfunction  $\varphi_n = \sin n\pi x$  has exactly  $n - 1$  simple zeros in  $(0, 1)$ .

Of course, the natural question is what would happen if we consider the sign-changing solutions of (1.1) under  $\kappa \neq 0$ ? It is the purpose of this paper to study the *higher eigenvalue* case for (1.1). More precisely, we shall show the following:

**Theorem 1.1** Assume that  $\kappa > 0$ . Let

$$S_n^\nu := \{u \in C_0^1[0, 1] \mid u \text{ has exactly } n - 1 \text{ simple zeros in } (0, 1), \text{ and } \nu u \text{ is positive near } 0\}.$$

for  $n \in \mathbb{N}$  and  $\nu \in \{+, -\}$ . Then

1. Problem (1.4) has a  $S_n^\nu$ -solution if and only if

$$8n^2 B^2 < \lambda < n^2 \pi^2, \quad B = \int_0^1 \frac{d\theta}{\sqrt{\theta^{-4} - 1}}. \quad (1.9)$$

2. The  $S_n^\nu$ -solution of (1.4) is unique if it exists. Subsequently, we denote it by  $u_\lambda$ .

3. All of the humps of  $u_\lambda$  are same, and the first hump is symmetric around  $\frac{1}{2n}$  for all  $\lambda$  satisfying (1.9).

4.  $u'_\lambda(0)$  is strictly decreasing in  $(8n^2 B^2, n^2 \pi^2)$ .

5.  $u_\lambda$  satisfies

$$\lim_{\lambda \downarrow 8n^2 B^2} \|u_\lambda\|_\infty = \frac{1}{2nB\sqrt{\kappa}},$$

and

$$\lim_{\lambda \uparrow n^2 \pi^2} \|u_\lambda\|_\infty = \lim_{\lambda \uparrow n^2 \pi^2} u_\lambda\left(\frac{1}{2n}\right) = 0.$$

6. The point-wise limit

$$u_{8n^2 B^2} := \lim_{\lambda \downarrow 8n^2 B^2} u_\lambda$$

provides us with a solution of (1.4) at  $\lambda = 8n^2B^2$  which is singular at  $t = \frac{j}{n}$ ,  $j = 0, 1, \dots, n$ , in the sense that

$$(-1)^j u'_{8B^2}(\frac{j}{n}) = \infty, \quad j = 0, 1, \dots, n. \quad (1.10)$$

**Theorem 1.2** Assume that  $\kappa < 0$ . Then

- (1) For  $n \in N$  and  $\nu \in \{+, -\}$ , (1.4) has a  $S_n^\nu$ -solution if and only if

$$n^2\pi^2 < \lambda < \infty. \quad (1.11)$$

- (2) The  $S_n^\nu$ -solution of (1.4) is unique if it exists. Subsequently, we denote it by  $u_\lambda$ .

- (3) All of the humps of  $u_\lambda$  are same, and the first hump is symmetric around  $\frac{1}{2n}$  for all  $\lambda$  satisfying (1.11).

- (4)  $u'_\lambda(0)$  is strictly increasing in  $(n^2\pi^2, \infty)$ .

- (5)  $u_\lambda$  satisfies

$$\lim_{\lambda \uparrow \infty} \|u_\lambda\|_\infty \leq \frac{1}{2n\sqrt{-\kappa}},$$

and

$$\lim_{\lambda \downarrow n^2\pi^2} \|u_\lambda\|_\infty = \lim_{\lambda \downarrow n^2\pi^2} u_\lambda(\frac{1}{2n}) = 0.$$

- (6) The point-wise limit

$$u_\infty := \lim_{\lambda \uparrow \infty} u_\lambda$$

satisfies

$$u'_\infty(\frac{j}{n}) := (-1)^j \frac{1}{\sqrt{-\kappa}}, \quad j \in \{0, 1, \dots, n\}.$$

Throughout this paper we use the following notations and conventions. Given a function  $u \in C[a, b]$  it is said that  $u \succ 0$  if  $u \geq 0$  but  $u \not\equiv 0$ . For any  $V \in C[a, b]$ , we shall denote by  $\sigma[-D^2 + V; (a, b)]$  the principal eigenvalue of  $-D^2 + V$  under homogeneous Dirichlet boundary conditions in the interval  $(a, b)$ . In particular,  $\sigma[-D^2, (0, 1)] = \pi^2$ .

Let  $E := C_0^1[0, 1]$  with the normal

$$\|u\|_1 := \max\{\|u\|_\infty, \|u'\|_\infty\}.$$

Then  $S_n^\nu$  is open in  $E$  for each  $n \in N$  and  $\nu \in \{+, -\}$ .

For the related results on the quasilinear problem (1.1) and its more general case (including the higher dimensional case), see D. Gilbarg and N. S. Trudinger [8], S.-Y. Cheng and S.-T. Yau [9], R. Bartnik and L. Simon [10], C. Bereanu, P. Jebelean and J. Mawhin [11], C. Bereanu, P. Jebelean, P. J. Torres [12-13], M. F. Bidaut-Véron and A. Ratto [14], I. Coelho, C. Corsato, F. Obersnel and P. Omari [15], R. López [16], J. Mawhin [17], A. E. Treibergs [18], A. Azzollini [19], H. Pan and R. Xing [20] and references therein.

The contents of this paper have been distributed as follows. In Section 2, we show that any nontrivial solution of (1.4) belongs to  $S_n^\nu$  for some  $n \in \mathbb{N}$  and  $\nu \in \{+, -\}$ , the  $S_n^\nu$ -solutions have the same humps and the first hump is symmetric around  $\frac{1}{2n}$ , and  $u$  is a  $S_n^\nu$  solution of (1.4) with  $\kappa > 0$  implies  $\lambda \in (0, n^2\pi^2)$ . Section 3 is devoted to show that (1.4) with  $\kappa > 0$  has a  $S_n^\nu$ -solution if and only if  $8n^2B^2 < \lambda < n^2\pi^2$ , and complete the proof of Theorem 1.1. In section 4, we state and prove some properties for the nontrivial solutions of (1.4) with  $\kappa < 0$ . Section 5 is devoted to show that (1.4) with  $\kappa < 0$  has a  $S_n^\nu$ -solution if and only if  $n^2\pi^2 < \lambda < \infty$ , and complete the proof of Theorem 1.2.

## 2 The properties of nontrivial solutions of (1.4) with $\kappa > 0$

As the linearization of (1.4) at  $(\lambda, u) = (\lambda, 0)$  is given by (1.2), by the local bifurcation theorem of Crandall and Rabinowitz [2,3], for each  $n \in \mathbb{N}$ , (1.1) possesses a curve of  $S_n^\nu$ -solutions emanating from  $(\lambda, u) = (\lambda, 0)$  at  $\lambda = n^2\pi^2$ . Actually,  $n^2\pi^2$  is the unique bifurcation value from  $u = 0$  to  $S_n^\nu$ -solutions of (1.4).

The next result shows that any nontrivial solution  $u$  of (1.4) cannot have a degenerate zero.

**Lemma 2.1** Let  $u$  be a nontrivial solution of (1.4) with  $\kappa \neq 0$  for some  $\lambda \in (0, \infty)$ . Then,  $u \in S_n^\nu$  for some  $\nu \in \{+, -\}$  and  $n \in \mathbb{N}$ .

**Proof.** Suppose on the contrary that

$$u(\tau) = u'(\tau) = 0$$

for some  $\tau \in [0, 1]$ . Then  $u$  is a solution of the initial value problem

$$\begin{cases} -u''(x) = \lambda u[1 + \kappa(u'(x))^2]^{3/2}, & 0 < x < 1, \\ u(\tau) = u'(\tau) = 0 \end{cases} \quad (2.1)$$

which implies that  $u(x) \equiv 0$  for  $x \in [0, 1]$ . This is a contradiction.  $\square$

The next result shows that this bifurcation is sub-critical.

**Lemma 2.2** Let us assume  $\kappa > 0$ . Let  $u$  be a  $S_n^\nu$ -solution of (1.1) for some  $\lambda \in (0, \infty)$ . Then  $0 < \lambda < n^2\pi^2$ . Moreover,

- (1) All the positive bumps of  $u$  have the same shape, and any such bump  $B$  satisfies
  - (i)  $B$  is symmetric about its mid-point  $m_B$ ;
  - (ii)  $u'$  is strictly decreasing on  $B$ , so  $B$  contains exactly one zero of  $u'$  at  $m_B$ ;
- (2) All the negative bumps of  $u$  have the same shape, and any such bump  $D$  satisfies
  - (i)  $D$  is symmetric about its mid-point  $m_D$ ;
  - (ii)  $u'$  is strictly increasing on  $D$ , so  $m_D$  contains exactly one zero of  $u'$  at  $m_D$ ;
- (3) The positive and negative bumps have the same shape.

**Proof.** Without loss of generality, we assume  $u \in S_n^+$ . Let

$$b := u'(0).$$

Then  $b > 0$ . Notice that  $u$  is also a nontrivial solution of the initial value problem

$$\begin{cases} -\left(\frac{u'(x)}{\sqrt{1 + \kappa(u'(x))^2}}\right)' = \lambda u(x), & 0 < x < 1, \\ u(0) = 0, \quad u'(0) = b. \end{cases} \quad (2.2)$$

Denote

$$v = u'. \quad (2.3)$$

Then (1.3) can be written in the form

$$\frac{v'}{(1 + \kappa v^2)^{3/2}} = \lambda u$$

and, hence

$$\frac{vv'}{(1 + \kappa v^2)^{3/2}} = \lambda uu'$$

or, equivalently,

$$\frac{d}{dx} \left( -\frac{1}{\kappa}(1 + \kappa v^2)^{-1/2} + \lambda \frac{u^2}{2} \right) = 0.$$

Therefore, for every  $x \in [0, 1]$ , we have that

$$\lambda \frac{\kappa}{2} u^2(x) - \frac{1}{\sqrt{1 + \kappa v^2(x)}} = -\frac{1}{\sqrt{1 + \kappa b^2}}. \quad (2.4)$$

It now follows from (2.4) that if  $b = 0$  then  $u \equiv 0$  on  $[0, 1]$ . Hence, if  $u \not\equiv 0$  then any zeros of  $u$  are simple, and  $u \neq 0$  at any zero of  $u'$ .

Let  $\tau \in [0, 1]$  be a zero of  $u$ . Suppose that  $u > 0$  on a maximal interval  $P$  with  $\tau$  is as a left end (similar arguments apply to maximal intervals  $N$  on which  $u < 0$ ). Then from the equation in (2.2),

$$\frac{u'(x)}{\sqrt{1 + \kappa(u'(x))^2}} = \frac{u'(\tau)}{\sqrt{1 + \kappa(u'(\tau))^2}} - \lambda \int_{\tau}^x u(s) ds, \quad x \in P. \quad (2.5)$$

Since the function

$$\psi(s) := \frac{s}{\sqrt{1 + \kappa s^2}}$$

is strictly increasing on  $(-\infty, \infty)$ , it follows from (2.5) that  $u'$  is strictly decreasing on  $P$ . Hence,  $P$  contains at most one zero of  $u'$ .

Now denote that  $P = (x_l, x_r)$ ,  $m := (x_r + x_l)/2$ . Then, by (2.4),

$$u(x_l) = u(x_r) = 0, \quad u'(x_l) = -u'(x_r) > 0. \quad (2.6)$$

By (2.6), it is easy to check that both  $u(x)$  and  $u(x_r + x_l - x)$  are solutions of initial value problem

$$\begin{cases} -\left(\frac{w'(x)}{\sqrt{1 + \kappa(w'(x))^2}}\right)' = \lambda w(x), & x_l < x < x_r, \\ w(x_l) = 0, & w'(x_l) = u'(x_l). \end{cases}$$

Hence, by the above uniqueness result, the solution curves on the intervals  $[x_l, m]$ ,  $[m, x_r]$  are symmetric, and  $u'(m) = 0$ . These results show that any positive (and negative) bumps have the properties (i) and (ii) described in the theorem.

To prove (3), we take  $n \geq 2$ . Let

$$0 = \tau_0 < \tau_1 < \cdots < \tau_n = 1$$

be the zeros of  $u$  in  $[0, 1]$ . Let us consider the initial value problem

$$\begin{cases} -v''(x) = \lambda u[1 + \kappa(v'(x))^2]^{3/2}, & 0 < x < 1, \\ v(0) = 0, & v'(0) = u'(0). \end{cases} \quad (2.7)$$

Since the equation in (1.3) is autonomous, it is easy to check that the function

$$v(x) := -u(x - \tau_1), \quad x \in [\tau_1, \tau_2]$$

is also a solution of the equation in (2.7). Now by the uniqueness results for the initial value problem (2.7),  $u \equiv v$  in  $[\tau_0, \tau_1]$ , and accordingly  $u \equiv v$  in  $[0, 1]$ , which implies the first positive bump and the first negative bump have the same shape.

To show that  $0 < \lambda < n^2\pi^2$ , multiplying the equation in (2.1) by  $\sin(n\pi x)$  and integrating in  $(0, \frac{1}{n})$ , we are lead to

$$\begin{aligned} n^2\pi^2 \int_0^{1/n} \sin(n\pi x)u(x)dx &= - \int_0^{1/n} \sin(n\pi x)u''(x)dx \\ &= \lambda \int_0^{1/n} \sin(n\pi x)u(x)V(x)dx \\ &> \lambda \int_0^{1/n} \sin(n\pi x)u(x)dx, \end{aligned}$$

where  $V := [1 + (u')^2]^{3/2}$ . Because  $V \succ 1$ , for as  $u = 0$  if  $V = 1$ . Therefore,  $\lambda < n^2\pi^2$ . The proof is complete.  $\square$

### 3 Interval of $\lambda$ in which (1.4) with $\kappa > 0$ has $S_n^\nu$ -solutions

Recall that

$$B := \int_0^1 \frac{d\theta}{\sqrt{\theta^{-4} - 1}}.$$

**Lemma 3.1** [1, Theorem 3.2.] The set

$$\mathcal{S}_1 := \{(\lambda, \|u_\lambda\|_\infty) \mid (\lambda, u_\lambda) \text{ is a symmetric positive solutions of (1.4)}\}$$

is a differentiable monotone curve, which connects  $(8B^2, \frac{1}{2B\sqrt{\kappa}})$  with  $(\pi^2, 0)$ . For each  $\lambda \in (8B^2, \pi^2)$ , the problem (1.4) admits a unique symmetric positive solution. Moreover, if  $u_\lambda$  stands for the unique symmetric positive solution of (1.4), then  $u'_\lambda(0)$  is strictly decreasing in  $(8B^2, \pi^2)$  and

$$u'_\lambda(0) = \infty.$$

Furthermore, the point-wise limit

$$u_{8B^2} := \lim_{\lambda \downarrow 8B^2} u_\lambda$$

provides us with a solution of (1.4) at  $8B^2$  which is singular at 0 and 1 in the sense that

$$u'_{8B^2}(0) = -u'_{8B^2}(1) = \infty.$$



In this section, we shall extend the above result to the case of nodal solutions.

**Theorem 3.2** For each  $n \in N$  and  $\nu \in \{+, -\}$ , the set

$$\mathfrak{S}_n^\nu := \{(\lambda, \|u_\lambda\|_\infty) \mid (\lambda, u_\lambda) \text{ is a } S_n^\nu\text{-solutions of (1.4)}\}$$

is a differentiable monotone curve, which connects  $(8n^2B^2, \frac{1}{2B\sqrt{\kappa}})$  with  $(\pi^2n^2, 0)$ . For each  $\lambda \in (8n^2B^2, n^2\pi^2)$ , the problem (1.4) admits a unique  $S_n^\nu$ -solution. Moreover, if  $u_\lambda$  stands for the unique  $S_n^\nu$ -solution of (1.4), then  $u'_\lambda(0)$  is strictly decreasing in  $(8n^2B^2, n^2\pi^2)$  and

$$\lim_{\lambda \downarrow 8n^2B^2} u'_\lambda(0) = \infty.$$

Furthermore, the point-wise limit

$$u_{8n^2B^2} := \lim_{\lambda \downarrow 8n^2B^2} u_\lambda$$

provides us with a solution of (1.4) at  $8n^2B^2$  which is singular at 0 and 1 in the sense that

$$(-1)^j u'_{8n^2B^2}\left(\frac{j}{n}\right) = \infty, \quad j \in \{0, 1, \dots, n\}.$$

**Proof.** By Lemma 2.2, to study the nodal solutions of (1.1) in  $\mathfrak{S}_n^\nu$ , it is enough to study the positive solutions of

$$\begin{cases} -\left(\frac{u'(x)}{\sqrt{1 + \kappa(u'(x))^2}}\right)' = \lambda u(x), & 0 < x < \frac{1}{n}, \\ u(0) = u\left(\frac{1}{n}\right) = 0. \end{cases} \quad (3.1)$$

The change of variable

$$u(x) = v(y), \quad y = xn, \quad 0 \leq x \leq \frac{1}{n} \quad (3.2)$$

transforms (3.1) into

$$\begin{cases} -\left(\frac{v'(y)}{\sqrt{1 + \tilde{\kappa}(v'(y))^2}}\right)' = \tilde{\lambda} v(y), & 0 < y < 1, \\ v(0) = v(1) = 0, \end{cases} \quad (3.3)$$

where we are denoting

$$\tilde{\kappa} := \kappa n^2, \quad \tilde{\lambda} := \frac{\lambda}{n^2}. \quad (3.4)$$

As (3.3) is of the same type as (1.4), by the analysis already done in Section 2, it becomes apparent that (3.1) possesses a positive symmetric solution around in  $\frac{1}{2n}$  if and only if

$$8B^2 < \tilde{\lambda} < \pi^2 \Leftrightarrow 8n^2B^2 < \lambda < n^2\pi^2. \quad (3.5)$$

Furthermore, we may use Lemma 3.1 to get the desired results.  $\square$

## 4 The properties of nontrivial solutions of (1.4) with $\kappa < 0$

As the linearization of (1.4) at  $(\lambda, u) = (\lambda, 0)$  is given by (1.2), by the local bifurcation theorem of Crandall and Rabinowitz [2,3], for each  $n \in \mathbb{N}$  and  $\nu \in \{+, -\}$ , (1.4) possesses a curve of  $S_n^\nu$ -solutions emanating from  $(\lambda, u) = (\lambda, 0)$  at  $\lambda = n^2\pi^2$ . Actually,  $n^2\pi^2$  is the unique bifurcation value from  $u = 0$  to  $S_n^\nu$ -solutions of (1.4).

The next result shows that this bifurcation is sup-critical in the case  $\kappa < 0$ .

**Lemma 4.1** Let  $\kappa < 0$ . Let  $u$  be a  $S_n^\nu$ -solution of (1.4) for some  $\lambda \in (0, \infty)$ . Then  $n^2\pi^2 < \lambda < \infty$ .

Moreover,

- (1) All the positive bumps of  $u$  have the same shape, and any such bump  $B$  satisfies
  - (i)  $B$  is symmetric about its mid-point  $m_B$ ;
  - (ii)  $u'$  is strictly decreasing on  $B$ , so  $B$  contains exactly one zero of  $u'$  at  $m_B$ ;
- (2) All the negative bumps of  $u$  have the same shape, and any such bump  $D$  satisfies
  - (i)  $D$  is symmetric about its mid-point  $m_D$ ;
  - (ii)  $u'$  is strictly increasing on  $D$ , so  $m_D$  contains exactly one zero of  $u'$  at  $m_D$ ;
- (3) The positive and negative bumps have the same shape.

**Proof.** Using the same method to prove Lemma 2.2, with obvious changes, we may get the that the humps of  $u$  have the same shape.

To show that  $n^2\pi^2 < \lambda < \infty$ , multiplying the equation in (2.1) by  $\sin(n\pi x)$  and integrating in  $(0, \frac{1}{n})$ , we are lead to

$$\begin{aligned} n^2\pi^2 \int_0^{1/n} \sin(n\pi x)u(x)dx &= - \int_0^{1/n} \sin(n\pi x)u''(x)dx \\ &= \lambda \int_0^{1/n} \sin(n\pi x)u(x)V(x)dx \\ &< \lambda \int_0^{1/n} \sin(n\pi x)u(x)dx, \end{aligned}$$

where  $V := [1 + \kappa(u')^2]^{3/2}$ . Because  $1 \succ V$ , for as  $u = 0$  if  $V \equiv 1$ . Therefore,  $\lambda > n^2\pi^2$ . The proof is complete.  $\square$

## 5 Interval of $\lambda$ in which (1.4) with $\kappa < 0$ has $S_n^\nu$ -solutions

We only determine the maximum interval of  $\lambda$  in which (1.4) with  $\kappa < 0$  has  $S_n^+$ -solutions since the other case can be treated similarly.

From Lemma 4.1,  $u \in S_n^+$ , the humps of  $u$  are same, and the first hump is symmetric around  $\frac{1}{2n}$ , and

$$u'(x) > 0 \text{ if } x \in [0, \frac{1}{2n}), \quad u'(\frac{1}{2n}) = 0, \quad u'(x) < 0 \text{ if } x \in (\frac{1}{2n}, \frac{1}{n}]. \quad (5.1)$$

So we only need to study the maximum interval of  $\lambda$  in which (1.4) with  $\kappa < 0$  has symmetric positive solutions in  $(0, \frac{1}{n})$ .

First, we consider the special case that  $u \in S_1^+$  and establish the following

**Theorem 5.1** The set

$$\mathfrak{S}_1^+ := \{(\lambda, \|u_\lambda\|_\infty) \mid (\lambda, u_\lambda) \text{ is a } S_1^+ \text{-solutions of (1.4)}\}$$

consists of a differentiable monotone curve in  $[\pi^2, \infty) \times [0, \frac{1}{2\sqrt{-\kappa}})$ . For each  $\lambda \in (\pi^2, \infty)$ , the problem (1.4) admits a unique  $S_1^+$ -solution. Moreover, if  $u_\lambda$  stands for the unique  $S_1^+$ -solution of (1.4), then  $u'_\lambda(0)$  is strictly increasing in  $(\pi^2, \infty)$ . Furthermore, the point-wise limit

$$u_\infty(t) := \lim_{\lambda \uparrow \infty} u_\lambda(t),$$

which satisfies

$$u'_\infty(0) = -u'_\infty(1) = \frac{1}{\sqrt{-\kappa}}.$$

To prove Theorem 5.1, we need some preliminaries.

Denote

$$u_0 = u(\frac{1}{2}), \quad v_0 = u'(0), \quad v = u'.$$

Thanks to (5.1) with  $n = 1$ , (1.5) can be written in the form

$$-\frac{v'}{(1 + \kappa v^2)^{3/2}} = \lambda u.$$

and hence

$$-\frac{vv'}{(1 + \kappa v^2)^{3/2}} = \lambda uu',$$

or, equivalently,

$$\frac{d}{dx} \left( -\frac{1}{\kappa}(1 + \kappa v^2)^{-1/2} + \lambda \frac{u^2}{2} \right) = 0.$$

Therefore, for every  $x \in [0, 1]$ , we have that

$$\lambda \frac{\kappa}{2} u^2(x) - \frac{1}{\sqrt{1 + \kappa v^2(x)}} = -\frac{1}{\sqrt{1 + \kappa v_0^2}} = \lambda \frac{\kappa}{2} u_0^2 - 1, \quad (5.2)$$

$$v(x) = u'(x) = \sqrt{\frac{1}{-\kappa}} \sqrt{1 - \left\{ 1 - \lambda \frac{\kappa}{2} (u_0^2 - u^2(x)) \right\}^{-2}} \quad \text{for all } x \in [0, \frac{1}{2}], \quad (5.3)$$

and

$$v(x) = u'(x) = -\sqrt{\frac{1}{-\kappa}} \sqrt{1 - \left\{ 1 - \lambda \frac{\kappa}{2} (u_0^2 - u^2(x)) \right\}^{-2}} \quad \text{for all } x \in [\frac{1}{2}, 1]. \quad (5.4)$$

Thus, necessarily

$$\begin{aligned} \frac{1}{2} &= \sqrt{-\kappa} \int_0^{1/2} \frac{u'(x)}{\sqrt{1 - \left\{ 1 - \lambda \frac{\kappa}{2} (u_0^2 - u^2(x)) \right\}^{-2}}} dx \\ &= \sqrt{-\kappa} \int_0^{u_0} \frac{1}{\sqrt{1 - \left\{ 1 - \lambda \frac{\kappa}{2} (u_0^2 - u^2) \right\}^{-2}}} du \\ &= \sqrt{-\kappa} \int_0^1 \frac{u_0}{\sqrt{1 - \left\{ 1 - \lambda \frac{\kappa}{2} u_0^2 (1 - \theta^2) \right\}^{-2}}} d\theta \end{aligned}$$

where  $\theta := \frac{u}{u_0}$ .

Equivalently,

$$\frac{1}{2} = J(\lambda, u_0), \quad (5.5)$$

where  $J : D(J) \rightarrow \mathbb{R}_+ := (0, \infty)$  is the function defined by

$$J(\lambda, \xi) = \sqrt{-\kappa} \int_0^1 \frac{\xi}{\sqrt{1 - \left\{ 1 - \lambda \frac{\kappa}{2} \xi^2 (1 - \theta^2) \right\}^{-2}}} d\theta \quad (5.6)$$

for all  $(\lambda, \xi) \in D(J)$ , where

$$D(J) = \left\{ (\lambda, \xi) \mid (\lambda, \xi) \in \mathbb{R}_+ \times \mathbb{R}_+ \right\}.$$

Owing to (5.3), it can be easily seen that the symmetric positive solutions of (1.4) are in one-to-one correspondence with the pairs  $(\lambda, u_0) \in D(J)$  satisfying (5.5).

The next result provides us with some important monotonicity properties of the function  $J(\lambda, \xi)$ .

**Proposition 5.2** (i) For every  $\xi > 0$ , the function  $\lambda \mapsto J(\lambda, \xi)$  is decreasing and

$$\lim_{\lambda \downarrow 0} J(\lambda, \xi) = \infty. \quad (5.7)$$

(ii) For every  $\lambda > 0$ , the function  $\varphi(\xi) := J(\lambda, \xi)$  satisfies  $\varphi'(\xi) > 0$ , and, hence, it is increasing.

**Proof.** The proof of (i) is straightforward and, hence, we omit its details. The proof of (ii) is more delicate. From (5.6), differentiating with respect to  $\xi$  and rearranging terms, we find that

$$\begin{aligned}\varphi'(\xi) &= \sqrt{-k} \int_0^1 \{1 - [1 - \lambda \frac{k}{2} \xi^2 (1 - \theta^2)]^{-2}\}^{-\frac{3}{2}} [1 - \lambda \frac{k}{2} \xi^2 (1 - \theta^2)]^{-3} \lambda k \xi^2 (1 - \theta^2) d\theta \\ &\quad + \sqrt{-k} \int_0^1 \{1 - [1 - \lambda \frac{k}{2} \xi^2 (1 - \theta^2)]^{-2}\}^{-\frac{1}{2}} d\theta \\ &= \sqrt{-k} \int_0^1 \{1 - [1 - \lambda \frac{k}{2} \xi^2 (1 - \theta^2)]^{-2}\}^{-\frac{3}{2}} \{[1 - \lambda \frac{k}{2} \xi^2 (1 - \theta^2)]^{-3} \lambda k \xi^2 (1 - \theta^2) \\ &\quad + 1 - [1 - \lambda \frac{k}{2} \xi^2 (1 - \theta^2)]^{-2}\} d\theta.\end{aligned}$$

Notice that  $k < 0$  and set

$$\alpha(\theta) := [1 - \lambda \frac{k}{2} \xi^2 (1 - \theta^2)]^{-1}, \quad 0 \leq \theta \leq 1. \quad (5.8)$$

Then

$$0 < \alpha(\theta) \leq 1 \quad \text{and} \quad 1 - \alpha^2(\theta) \geq 0 \quad \forall \theta \in [0, 1]$$

and

$$\varphi'(\xi) = \sqrt{-k} \int_0^1 (1 - \alpha^2)^{\frac{3}{2}} [\alpha^3 \lambda k \xi^2 (1 - \theta^2) + (1 - \alpha^2)] d\theta.$$

On the other hand, according to (5.8), it becomes apparent that

$$\alpha \lambda k \xi^2 (1 - \theta^2) = 2(\alpha - 1)$$

and, hence,

$$\begin{aligned}\varphi'(\xi) &= \sqrt{-k} \int_0^1 (1 - \alpha^2)^{\frac{3}{2}} [2\alpha^2(\alpha - 1) + (1 - \alpha^2)] d\theta \\ &= 2\sqrt{-k} \int_0^1 (1 - \alpha^2)^{\frac{3}{2}} (1 - \alpha)^2 (\alpha + \frac{1}{2}) d\theta > 0.\end{aligned}$$

The proof is complete. □

**Proof of Theorem 5.1.** By the fact

$$-\kappa v_0^2 < 1.$$

This together with the symmetric property of  $u$  imply that

$$\xi < \frac{1}{2\sqrt{-\kappa}}.$$

From Proposition 5.2, there exists an interval  $(0, \rho) \subseteq (0, \frac{1}{2\sqrt{-\kappa}})$ , such that for every  $\xi \in (0, \rho)$ , there exists a unique  $\lambda(\xi) > 0$  satisfying

$$J(\lambda(\xi), \xi) = \frac{1}{2},$$

i.e.

$$\frac{1}{2} = \frac{1}{\sqrt{\lambda(\xi)}} \int_0^1 \frac{1 - \lambda \frac{\kappa}{2} \xi^2 (1 - \theta^2)}{\sqrt{1 - \theta^2 - \lambda \frac{\kappa}{4} \xi^2 (1 - \theta^2)^2}} d\theta$$

and, therefore, since  $\pi^2 < \lambda(\xi)$ , letting  $\xi \rightarrow 0$ , we get

$$\frac{1}{2} = \frac{1}{\sqrt{\lambda(0)}} \int_0^1 \frac{d\theta}{\sqrt{1 - \theta^2}} = \frac{1}{\sqrt{\lambda(0)}} \frac{\pi}{2}.$$

Hence,  $\lambda(0) := \lim_{\xi \rightarrow 0} \lambda(\xi) = \pi^2$ .

By the monotonicity and continuity of  $J$ ,  $\xi \rightarrow \lambda(\xi)$  must be continuous and strictly increasing. The differentiability of  $\mathfrak{S}_1^+$  is a direct consequence from Proposition 5.2(ii) and the implicit function theorem.

The function  $u'_\lambda(0)$  is strictly increasing for  $\lambda \in (\pi^2, \infty)$  is an immediate consequence of the fact  $\xi \rightarrow \lambda(\xi)$  is strictly increasing in  $(0, \rho)$  and the relation

$$\lambda \frac{\kappa}{2} u^2(x) - \frac{1}{\sqrt{1 + \kappa v^2(x)}} = -\frac{1}{\sqrt{1 + \kappa v_0^2}} = \lambda \frac{\kappa}{2} \xi^2 - 1.$$

On the other hand, let

$$\lim_{\lambda \uparrow \infty} u_\lambda\left(\frac{1}{2}\right) = u_\infty\left(\frac{1}{2}\right) =: \xi^*.$$

Then  $\xi^* > 0$ . Combining this with the fact  $-\frac{1}{\sqrt{1 + \kappa v_0^2}} = \lambda \frac{\kappa}{2} \xi^2 - 1$  and letting  $\lambda \rightarrow \infty$ , we get

$$\lim_{\lambda \uparrow \infty} u'_\lambda(0) = u'_\infty(0) = \frac{1}{\sqrt{-\kappa}},$$

and this completes the proof.  $\square$

Now, we are in the position to prove Theorem 1.2.

**Proof of Theorem 1.2.** By Lemma 4.1, to study the set of  $S_n''$ -solutions of (1.4), it is enough to study the set of positive solutions of

$$\begin{cases} -\left(\frac{u'(x)}{\sqrt{1 + \kappa(u'(x))^2}}\right)' = \lambda u(x), & 0 < x < \frac{1}{n}, \\ u(0) = u\left(\frac{1}{n}\right) = 0. \end{cases} \quad (5.9)$$

The change of variable

$$u(x) = v(y), \quad y = xn, \quad 0 \leq x \leq \frac{1}{n}, \quad (5.10)$$

transforms (5.9) into

$$\begin{cases} -\left(\frac{v'(y)}{\sqrt{1 + \tilde{\kappa}(v'(y))^2}}\right)' = \tilde{\lambda} v(y), & 0 < y < 1, \\ v(0) = v(1) = 0, \end{cases} \quad (5.11)$$

where we are denoting

$$\tilde{\kappa} := \kappa n^2, \quad \tilde{\lambda} := \lambda/n^2. \quad (5.12)$$

Notice that all of the conclusions of Theorem 5.1 for (5.11) are still valid for (5.9) via the transformation (5.10). Thus, Theorem 1.2 can be deduced from (5.12) and Theorem 5.1.  $\square$

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